SOLITONS AND THE INVERSE SCATTERING TRANSFORM

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to Carol and Enid

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Preface

In this book we have attempted to bring together much of the work that has been accomplished in the field which we loosely term: Solitons and the Inverse Scattering Transform. Usually, our procedure has been to explain the basic mathematical ideas by means of examples rather than by considering the most general situation. Attempts have been made to incorporate many of the important research papers into our bibliography. Unfortunately we are almost certain to have missed some relevant research articles. For this we apologize. Similarly, due to time considerations, we have not been able to include some of the very recent advances in this field. It should be remarked that this area of study is continuing to develop in a vigorous manner.

We are indebted to a number of people who have helped to make this book possible. Naturally, this includes all of the people whose research in this subject has influenced our own. Special thanks go to Martin Kruskal, who has profoundly influenced our point of view; to David Kaup and Alan Newell, who made up the other half of "AKNS"; to Junkichi Satsuma and Guido Sandri, who made a number of useful comments and suggestions while we were preparing the manuscript; and to David Benney, who introduced the subject of nonlinear waves to one of us (MJA). Our own research in this area was partially funded by the Air Force Office of Scientific Research, the Army Research Office, the National Science Foundation and the Office of Naval Research (Mathematics and Fluid Dynamics programs). We are grateful to our technical monitors at all of these agencies for their support and encouragement. Our secretaries, Rita Gruda, Barbara Hawk, Ninon Hutchinson, Marilyn Kreizman, Cindy Martin and Celia Woodson, were given the unpleasant task of transforming many pages of unreadable scrawl into a legible manuscript. Finally we are grateful to our wives, Enid Ablowitz and Carol Segur, who put up with many late hours and working weekends.

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Prologue

The basic theme of this book can be stated quite simply: *Certain nonlinear problems have a surprisingly simple underlying structure, and can be solved by essentially linear methods.* Typically, these problems are in the form of *evolution equations*, which describe how some variable (or set of variables) evolves in time from a given initial state. The equations may take a variety of forms, including partial differential equations, differential-difference (discrete space, continuous time), partial difference (discrete time and space), integro-differential, as well as coupled ordinary differential equations (of finite order). What is surprising is that even though these problems are nonlinear, one may obtain the general solution that evolves from arbitrary initial data (within an appropriate class) without approximation. It is perhaps equally surprising that some of these exactly solvable problems arise naturally as models of physical phenomena. These applications have helped to generate interest in the subject.

Several viewpoints about these exactly solvable problems are common. One of them identifies the general solution of an appropriate initial value problem as *the* objective of the analysis. This solution is obtained by the Inverse Scattering Transform (IST), which is described in detail in Chapters 1 and 2. It can be viewed as a generalization of the Fourier transform, by which linear problems may be solved.

The problems in question have such a rich structure that they may be considered from several other viewpoints, which may be rather unrelated to IST. Some of these other perspectives are examined in Chapter 3. Many of these are more useful if one is primarily interested in special solutions, such as solitons, rather than in the general solution of an initial value problem. A number of physical applications are discussed in detail in Chapter 4.

The value of IST is that one treats nonlinear problems by essentially linear methods. This value is marginal, of course, unless one is already familiar with the methods and results of linear theory. Because of the fundamental role played by linear theory, we have included an extensive appendix which deals with linear problems and their solutions. These serve as useful guides against which to compare the corresponding solutions of the nonlinear problems that are the subject of this book.

Before we plunge in, here is our opinion regarding the order in which the book should be read. The Appendix contains material that is preliminary, although not necessarily trivial. It will be most useful to those unfamiliar with Fourier transform methods if it is read first. Because it is introductory, a substantial set of fairly straightforward exercises is included at the end of the Appendix.

Chapter 1 is fundamental. Later chapters often build on the material in this chapter, and refer back to it. We recommend that all of Chapter 1 be read.

Many avenues are available after Chapter 1. Chapters 2, 3 and 4 depend on Chapter 1, but not particularly on each other. They may be read in any order desired. To a lesser extent, the sections within each chapter may be considered independent of each other as well. This permits the reader with a specialized interest to gain access to his/her material relatively quickly.

Finally, a word about the exercises. These cover a range of difficulty, from merely filling in some missing steps, to research problems whose answers, to our knowledge, are open. Usually the wording of the problems identifies to the reader the ones that are open.

Chapter 1

The Inverse Scattering Transform on the Infinite Interval

1.1. Introduction. In 1965 Zabusky and Kruskal discovered that the pulselike solitary wave solution to the Korteweg–deVries (KdV) equation had a property which was previously unknown: namely, that this solution interacted "elastically" with another such solution. They termed these solutions *solitons*. Shortly after this discovery, Gardner, Greene, Kruskal and Miura (1967), (1974) pioneered a new method of mathematical physics. Specifically, they gave a method of solution for the KdV equation by making use of the ideas of direct and inverse scattering. Lax (1968) considerably generalized these ideas, and Zakharov and Shabat (1972) showed that the method indeed worked for another physically significant nonlinear evolution equation, namely, the nonlinear Schrödinger equation. Using these ideas Ablowitz, Kaup, Newell and Segur (1973*b*) and (1974) developed a method to find a rather wide class of nonlinear evolution equations solvable by these techniques. They termed the procedure the Inverse Scattering Transform (IST).

This monograph is devoted to this subject: i.e., to solitons and IST. There have been numerous developments in this area, which has aroused considerable interest among mathematicians, physicists and engineers. We hope that by capturing many of the main ideas and putting them into one location, we will be helpful to both beginners and the "pros" in the field. The main difficulty in doing this comes from the vigor with which the field has and is (at this time) continuing to develop.

Some review articles[1] and some collected works[2] on the subject are available. At the time of writing, there are not any other monographs extant on this topic, but we expect that this state of affairs will undoubtedly change quickly.[3]

[1] See, for example, Scott, Chu and McLaughlin (1973), Miura (1976), Ablowitz (1978) and Makhankov (1979).

[2] See, for example, Newell (1974*a*), Miura (1974), Moser (1975*b*), Calogero (1978*a*) and Lonngren and Scott (1978).

[3] In fact, by the time galley proofs for this monograph were received, both Zakharov, Manakov, Novikov and Pitayevsky (1980) and Lamb (1980) had appeared.

The study of solitary waves began with the observations by J. Scott Russell (1838), (1844) over a century ago. Russell, an experimentalist, first observed a solitary wave while riding on horseback beside a narrow barge channel. When the boat he was observing stopped, Russell noted that it set forth

> a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed $\cdots$. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon. (Russell (1838)).

This observation inspired Russell to initiate an extensive experimental investigation of water waves. He divided all of water waves into two classes, the "great primary wave of translation" (which would eventually be called a solitary wave), and all other waves, which "belong to the *second or oscillatory order of waves*;" the latter waves "are not of the first order" (Russell (1838)). Clearly he regarded the solitary waves as being of primary importance and concentrated most of his attention on them. Among his many results, we should note particularly the following.

1. Solitary waves, which are long (shallow water) waves of permanent form, *exist*. This is undoubtedly his most important result.

2. The speed of propagation of a solitary wave in a channel of uniform depth is given by

$$v=\sqrt{g(h+\eta)},$$

η "being the height of the crest of the wave above the plane of repose of the fluid, h the depth throughout the fluid in repose, and g the measure of gravity" (Russell (1844)). Considering the accuracy of the experimental equipment available to him, this result is somewhat remarkable.

Russell found that no mathematical theory available at the time predicted a solitary wave, but noted that

> it was not to be supposed that after its existence had been discovered and its phenomena determined, endeavors would not be made to reconcile it with previously existing theory, or in other words, to show how it ought to have been predicted from the known general equations of fluid motion. In other words, it now remained to the mathematician to predict the discovery after it had happened; i.e., to give an *a priori* demonstration *a posteriori*. (Russell (1844)).

Russell seems to have been particularly contemptuous of Airy, who published a theory of long waves of small but finite amplitude in his *Tides and Waves* (1845). This theory is summarized in Lamb (1932, §§ 175 and 187), who states "when the elevation η is not small compared with the mean depth, h, waves, even in an uniform-canal of rectangular section, are no longer propagated without change of type." Thus, Airy concluded that solitary waves of permanent form do not exist! He also found an approximate formula for the wave speed,

$$v=\sqrt{gh}\left(1+\frac{1}{2}\frac{\eta}{h}\right),$$

which agrees with Russell's result to first order in (η/h). From this, Airy decided that "we think ourselves fully entitled to conclude from these experiments [i.e., Russell's] that the theory [Airy's] is entirely supported." Russell described Airy's conclusion as "completely the opposite of that to which we should be led on the same grounds."

This controversy raged on for another 50 years before it was finally resolved by Korteweg and de Vries (1895). They derived an equation (now known as the Korteweg–deVries, or KdV equation), which governs moderately small, shallow-water waves. Their equation had permanent wave solutions, including solitary waves.

Boussinesq (1871), (1872), also derived a nonlinear evolution equation governing such long waves. Both Boussinesq (1871), (1872) and Rayleigh (1876) obtained solitary wave solutions.

As Miura (1976) points out, despite this early work, apparently no new applications of the equation derived by Korteweg and deVries were discovered until 1960. Then, while studying collision-free hydromagnetic waves, Gardner and Morikawa (1960) also derived the Korteweg–deVries equation.

The physical problem which motivated the recent discoveries related to the KdV equation was the Fermi–Pasta–Ulam (FPU) problem (1955). In 1914 Debye suggested that the finiteness of the thermal conductivity of an anharmonic lattice is due to its nonlinearity. This led Fermi, Pasta and Ulam to undertake a numerical study of a one-dimensional anharmonic lattice. They felt that, due to the nonlinear coupling, any smooth initial state would eventually relax to an equipartition of energy among the various degrees of freedom of the system.

The model they considered consisted of identical masses connected to their nearest neighbors by nonlinear springs with the force law $F(\Delta)=-K(\Delta+\alpha\Delta^2)$. The equations of motion are

$$\frac{m}{K}y_{i,tt}=(y_{i+1}+y_{i-1}-2y_i)+\alpha[(y_{i+1}-y_i)^2-(y_i-y_{i-1})^2],\qquad i=1,2,\cdots,N-1,\quad y_0=y_N=0, \tag{1.1.1}$$

with a typical initial condition of $y_i(0)=\sin i\pi/N$, $y_{it}(0)=0$ (typically N was taken to be 64). Here y_i measures the displacement of the ith mass from equilibrium.

According to Fermi, Pasta and Ulam (1955),

> the results of our computations show features which were, from the beginning, surprising to us. Instead of a gradual, continuous flow of energy from the first mode to the higher modes, $\cdots$ the energy is exchanged, essentially, among only a certain few. $\cdots$ There seems to be little if any, tendency towards equipartition of energy among all degrees of freedom at a given time. In other words, the systems certainly do not show mixing.

In order to understand this phenomenon, Kruskal and Zabusky (1963) considered a continuum model. Calling the length between springs h, $t' = \omega t$ $(\omega = \sqrt{K/m})$, $x' = x/h$ with $x = ih$, and expanding $y_{i\pm 1}$ in Taylor series, reduces (1.1.1) to (dropping the primes)

$$y_{tt} = y_{xx} + \varepsilon y_x y_{xx} + \frac{h^2}{12} y_{xxxx} + O(\varepsilon h^2, h^4), \tag{1.1.2}$$

where $\varepsilon = 2\alpha h$. A further reduction is possible if we look for an asymptotic solution of the form (unidirectional waves)

$$y \sim \phi(X, T), \quad X = x - t, \quad T = \frac{\varepsilon t}{2},$$

whereupon (1.1.2) gives

$$\phi_{XT} + \phi_X \phi_{XX} + \delta^2 \phi_{XXXX} + O\left(h^2, \frac{h^4}{\varepsilon}\right) = 0, \tag{1.1.3}$$

where $\delta^2 = h^2/12\varepsilon$. By setting $u = \phi_X$, (1.1.3) is reduced to an equation directly related to that originally discovered by Korteweg and deVries (1895):

$$u_T + u u_X + \delta^2 u_{XXX} = 0. \tag{1.1.4}$$

Kruskal and Zabusky computed (1.1.4) typically with sinusoidal initial conditions. With δ^2 taken small, the slope of the initial function steepens until the third derivative terms become important. At this stage the solution develops an oscillatory structure of a definite form. The oscillations interact in a very definite and surprising way, which we will discuss presently. The process of trying to understand this phenomenon is what led to our present understanding of the properties and solutions of the KdV equation. (Interestingly Lax and Levermore (1979) have reinvestigated (1.1.4) with δ^2 small.)

Hereafter we shall work with the KdV equation in the following form:

$$K(u) = u_t + 6uu_x + u_{xxx} = 0. \tag{1.1.5}$$

Equation (1.1.5) is equivalent to (1.1.4) upon a scale change (note that any constant coefficient may be put in front of each of the three terms by suitably scaling the independent and dependent variables).

It was clear to Kruskal and Zabusky (and was well known) that KdV had a special permanent wave solution, the solitary wave,

$$u = 2k^2 \operatorname{sech}^2 k(x - 4k^2 t - x_0), \tag{1.1.6}$$

where k and x_0 are constant. Note that the velocity of this wave, $4k^2$, is proportional (by a factor of 2 here) to the amplitude, $2k^2$. What was not clear to previous researchers, and what is so surprising, is the way these waves interact with each other elastically. Indeed, in trying to understand the nature of the oscillations discussed above, Zabusky and Kruskal discovered the following. Suppose that at time $t = 0$, two such waves as (1.1.6) are given, well separated and with the smaller to the right. Then after a sufficient time the waves overlap and interact (the bigger one catches up). Following the process still longer, the bigger one separates from the smaller, and eventually (asymptotically) regains its initial shape and hence velocity. The only effect of the interaction is a phase shift; i.e., the center of each wave is at a different position than where it would have been if it had been traveling alone (see Fig. 1.1).

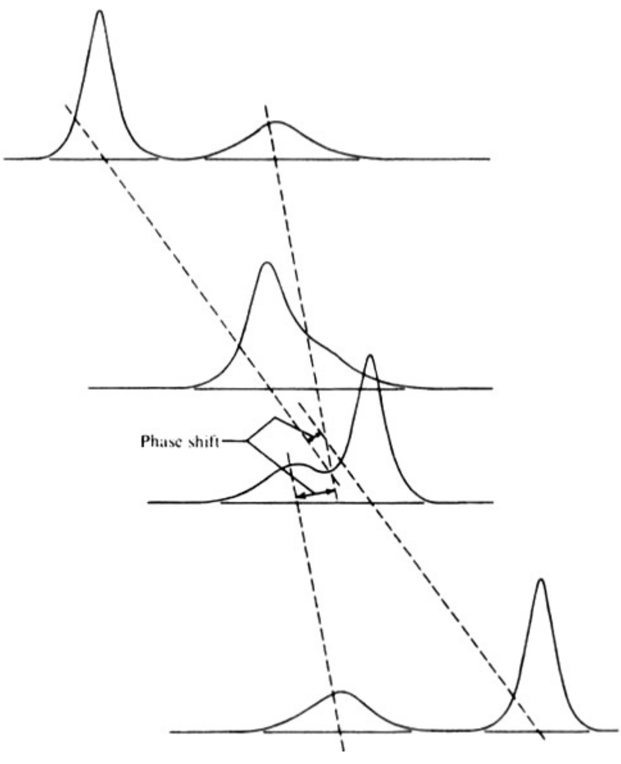

FIG. 1.1. *A typical interaction of two solitons at succeeding times.*

Because of the analogy with particles, Zabusky and Kruskal referred to these special waves as *solitons*. We shall follow their lead and refer to any localized nonlinear wave which interacts with another (arbitrary) local disturbance and always regains asymptotically its exact initial shape and velocity (allowing for a possible phase shift) as a soliton.

We refer to waves which interact inelastically as solitary waves. We also note that there are many different working definitions in the literature of what is, and what is not, a soliton. For our purpose the above definition is adequate.

In attempts to understand the initial onset of the oscillations in the numerical calculation of (1.1.4), the question of a "reversible" shock arose. A shock requires jump conditions; hence the question of jump conditions and conservation laws arose naturally. (A conservation law is an equation of the form $\partial T/\partial t + \partial F/\partial x = 0$, where T is called the density and F the flux, by analogy with fluid flow.) Early on, four conservation laws were obtained, Miura subsequently discovered a few more (Miura (1976)), and it was conjectured that there were an infinite number.

After studying these conservation laws, and those associated with a completely new evolution equation (which is commonly called the modified KdV equation or mKdV)

$$M(v) = v_t - 6v^2 v_x + v_{xxx} = 0, \tag{1.1.7}$$

Miura (1968) discovered the following transformation. If v is a solution of (1.1.7), then

$$u = -(v^2 + v_x) \tag{1.1.8}$$

is a solution of (1.1.5). Specifically,

$$K(u) = -\left(2v + \frac{\partial}{\partial x}\right) M(v). \tag{1.1.9}$$

Because of the operator in the right-hand side of (1.1.9), the transformation is single-valued in one direction only.

It was the transformation (1.1.9) that led to the other important results related to the KdV equation. Originally, (1.1.9) was the basis of a proof that the KdV equation indeed had an infinite number of conserved quantities, (Miura, Gardner and Kruskal (1968)). The basic idea is as follows. Since the KdV equation is Galilean invariant, the transformation

$$\begin{aligned} x' &= x + \frac{6}{\varepsilon^2} t, \qquad t' = t, \\ u(x,t) &= u'(x',t') - \frac{1}{\varepsilon^2} \end{aligned} \tag{1.1.10a}$$

leaves the KdV equation invariant, whereas setting

$$v(x, t) = -\varepsilon w(x', t') + \frac{1}{\varepsilon} \tag{1.1.10b}$$

transforms the mKdV equation into

$$w_{t'} + \frac{\partial}{\partial x'}(3w^2 + 2\varepsilon^2 w^3 + w_{x'x'}) = 0. \tag{1.1.10c}$$

Clearly $\int_{-\infty}^{\infty} w\, dx'$ is a conserved quantity of (1.1.10c). Similarly, from the Miura transformation (1.1.8), (1.1.10a, b) yield

$$u' = 2w + \varepsilon w_{x'} - \varepsilon^2 w^2. \tag{1.1.10d}$$

Thinking of $\varepsilon \ll 1$, we may solve (1.1.10d) recursively for w as a function of u', i.e.,

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w^2 + \cdots = \frac{u'}{2} - \frac{\varepsilon}{4} u'_{x'} + \frac{\varepsilon^2}{4}\left(\frac{u'_{x'x'}}{2} + u'^2\right) + \cdots. \tag{1.1.10e}$$

Hence, (1.1.10e) allows us to obtain an infinite number of conserved quantities. Later, in § 1.6, we shall give alternative proofs of the fact that KdV, mKdV, etc., have an infinite number of conserved quantities (or densities). Moreover, it can be shown that the even ones are nontrivial (i.e., not perfect derivatives).

The most significant result of all, however, was the development of a new method in mathematical physics, the Inverse Scattering Transform (IST). It too was motivated by (1.1.8). Note that (1.1.8) may be viewed as a Riccati equation for v in terms of u; the well-known transformation $v = \Psi_x/\Psi$ linearizes (1.1.8), yielding

$$\Psi_{xx} + u\Psi = 0.$$

Since the KdV equation is Galilean-invariant, and to be as general as possible, Miura, Gardner and Kruskal (1968) considered

$$\Psi_{xx} + (\lambda + u)\Psi = 0. \tag{1.1.11}$$

It turns out that this equation provides an implicit linearization of the KdV equation. Indeed, (1.1.11) is not an insignificant equation itself. It is the time-independent Schrödinger equation of quantum mechanics.

Gardner, Greene, Kruskal and Miura (1967), (1974) first discovered the method of solution of KdV by employing (1.1.11). Although we deviate from their original procedure, the ideas are of course similar. We postulate an associated time evolution equation,

$$\Psi_t = A\Psi + B\Psi_x, \tag{1.1.12}$$

where A, B are scalar functions independent of Ψ (note that this is the most general local, linear form of time dependence). We find that, if the KdV equation (1.1.5) is satisfied and if we choose

$$A = u_x, \qquad B = 4\lambda - 2u, \tag{1.1.13}$$

then the eigenvalues are invariant in time, i.e., $\lambda_t = 0$. In fact, the reader can verify that forcing the compatibility condition $\Psi_{txx} = \Psi_{xxt}$ yields

$$[K(u) + \lambda_t]\Psi = 0. \tag{1.1.14}$$

Hence if $K(u) = 0$, then $\lambda_t = 0$. In § 1.2 we shall give a deductive procedure for finding A, B. We will show that there are infinitely many equations associated with (1.1.11) in this way, with different A, B.

In subsequent sections we shall discuss in detail how the results (1.1.11)–(1.1.14) can be used to reconstruct potentials $u(x, t)$, given $u(x, t = 0)$. The method is somewhat sophisticated, and applies to a number of physically interesting evolution equations. The results in this field apply to a variety of physical problems, as discussed in Chapter 4. Moreover the mathematics used is also quite broad, ranging from classical analysis to differential geometry to algebra and to algebraic geometry (see also Chapter 3).

1.2. Second order eigenvalue problems and related solvable partial differential equations. As mentioned briefly in § 1.1, the inverse scattering transform (IST) was first developed and applied to the Korteweg–deVries (KdV) equation and its higher order analogues by Gardner, Greene, Kruskal and Miura (1967), (1974). At that time and shortly thereafter it was by no means clear if the method would apply to other physically significant nonlinear evolution equations. However, Zakharov and Shabat (1972) showed that the method was not a fluke. Using a technique first introduced by Lax (1968) they showed that the nonlinear Schrödinger equation

$$iq_t = q_{xx} + \kappa q^2 q^*, \qquad \kappa > 0 \tag{1.2.1}$$

is related to a certain linear scattering problem. Applying direct and inverse scattering ideas, they were able to solve (1.2.1) given initial values $q(x, 0)$ that decayed sufficiently rapidly as $|x| \to \infty$. Shortly thereafter, Wadati (1972), using these ideas, gave a method of solution for the modified Korteweg–deVries (mKdV) equation

$$q_t + 6q^2 q_x + q_{xxx} = 0, \tag{1.2.2}$$

and Ablowitz, Kaup, Newell and Segur (1973*a*) did the same for the "sine-Gordon" equation

$$u_{xt} = \sin u. \tag{1.2.3}$$

These results already showed the power and versatility of IST to solve certain physically interesting nonlinear PDE's.

Then Ablowitz, Kaup, Newell and Segur (1973*b*), (1974) developed procedures which, given a suitable scattering problem, allow one to derive the nonlinear evolution equations solvable by IST with that scattering problem. For example, it turns out that the KdV, modified KdV, nonlinear Schrödinger, and sine-Gordon equations can all be shown to be related to one master eigenvalue problem.

We begin by briefly considering the essential ideas behind Lax's (1968) approach. Consider two operators L, M, where L is the operator of the spectral problem and M is the operator of an associated time evolution equation

$$Lv = \lambda v, \tag{1.2.4a}$$

$$v_t = Mv. \tag{1.2.4b}$$

Associated with KdV is the Schrödinger scattering problem (1.1.11). Hence, in this case $L = \partial_x^2 + u(x, t)$.

Taking the time derivative of (1.2.4a) and assuming $\lambda_t = 0$, we have $L_t v + Lv_t = \lambda v_t$. Substitution of (1.2.4b) yields a condition which is necessary for (1.2.4a,b) to be compatible:

$$L_t + [L, M] = 0, \tag{1.2.4c}$$

where $[L, M] = LM - ML$(the commutator of L and M). Equation (1.2.4c) contains a nonlinear evolution equation if L and M are *correctly* chosen. Given L, Lax (1968) indicates how to construct an associated M so as to make (1.2.4c) nontrivial.

The difficulties with this method are that (a) one must "guess" a suitable L and then find an M in order to satisfy (1.2.4a, b) and (b) it is often awkward to work with differential operators (e.g., sine-Gordon (1.2.3)). As an alternative Ablowitz, Kaup, Newell and Segur (1974) proposed a technique which, very generally, can be formulated as follows. Consider two linear equations

$$\mathbf{v}_x = X\mathbf{v}, \tag{1.2.5a}$$

$$\mathbf{v}_t = T\mathbf{v}, \tag{1.2.5b}$$

where $\mathbf{v}$ is an n-dimensional vector and X, T are $n \times n$ matrices. Then cross differentiation (i.e., taking $\partial/\partial t$ (1.2.5a), $\partial/\partial x$ (1.2.5b) and setting them equal) yields

$$X_t - T_x + [X, T] = 0. \tag{1.2.6}$$

This is, in essence, the equivalent of (1.2.4c). It turns out that, given X, there is a simple deductive procedure to find a T such that (1.2.6) contains a nonlinear evolution equation. In order for (1.2.6) to be effective, the associated operator X should have a parameter which plays the role of an eigenvalue, say ζ, and obeys $\zeta_t = 0$. Moreover, a complete solution to the associated nonlinear evolution equation on the infinite interval can be found when the